

Conservative General Relativistic Radiation Hydrodynamics in Spherical Symmetry and Comoving Coordinates

Matthias Liebendörfer^{1,2}, Anthony Mezzacappa², and Friedrich-Karl Thielemann^{2,3}

¹ *Department of Physics and Astronomy, University of Tennessee, Knoxville, TN 37996-1200*

² *Physics Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831-6354*

³ *Department of Physics and Astronomy, University of Basel,
Klingelbergstrasse 82, CH-4056 Basel Switzerland*

The description of general relativistic radiation hydrodynamics in spherical symmetry is presented in natural coordinate choices. For hydrodynamics, comoving coordinates are chosen and the momentum phase space for the radiation particles is described in comoving frame four momenta. We also investigate a description of the momentum phase space in terms of particle impact parameter and energy at infinity and derive a simple approximation to the general relativistic Boltzmann equation. Further developed are, however, the exact equations in comoving coordinates since the description of the interaction between matter and radiation particles is best described in the closely related orthonormal basis comoving with the fluid elements. We achieve a conservative and concise formulation of radiation hydrodynamics that is well suited for numerical implementation by a variety of methods. The contribution of radiation to the general relativistic jump conditions at shock fronts is discussed and artificial viscosity is consistently included in the derivations in order to support approaches relying on this option.

I. INTRODUCTION

Astrophysical knowledge is gained by the observation of luminous objects and the search for scenarios that explain the observations based on accepted physics. Often, numerical simulations are required to determine if a detailed scenario really produces the given observation. In supernova theory, for example, the transition from analytical considerations of core collapse and explosion (Colgate and Johnson [1]) to numerical simulations (Colgate and White [2]) was made very early, when computers became available. General relativistic simulations of core collapse followed immediately (May and White [3]). These simulations were based on the newly derived Einstein equations in spherical symmetry (Misner and Sharp [4]) that served as a basis for many later investigations, including also the present paper. In comoving coordinates, general relativistic radiation transport was first formulated by Lindquist [5]. Here, we reformulate general relativistic radiation hydrodynamics in light of modern numerical algorithms requiring conservation laws, and hope to reveal some of the beauty in the spherically symmetric case that may have remained hidden.

However, comoving coordinates are only one possible choice out of a variety of 3+1 decompositions enabled by the covariance of general relativity in four-dimensional space-time (Arnowitt, Deser, and Misner [6]; Smarr and York [7]). The Boltzmann transport equation is usually split into a left-hand side and a right-hand side. The left-hand side is the directional derivative of the distribution function along trajectories of free particle propagation. This derivative is equated to the changes in the distribution function due to collisions. Thus, the right-hand side accounts for changes in the radiation particle distribution function owing to particles that are created, annihilated, or scattered into new states in the momentum phase space.

After the choices of a 3+1 decomposition and a basis in the momentum phase space have been made, the directional derivative along the phase flow can be expressed in terms of partial derivatives of the distribution function with respect to the coordinates. The complexity of the left-hand side as well as the complexity of the collision term depend on the coordinate choice. A general discussion of radiation transport in spherically symmetric 3+1 decomposition has been provided by Mezzacappa and Matzner [8]. Maximal slicing was chosen and the particle distribution function was described by the four momenta measured in the frame of an observer comoving with the matter. This choice eases the evaluation of angle dependent cross sections in the collision term (Mihalas [9], Mezzacappa and Bruenn [10]). Schinder and Bludman [11] chose polar slicing in the 3+1 decomposition and a tangent ray approach in the description of the momentum phase space. This choice avoids partial derivatives with respect to the momentum space variables on the left-hand side of the Boltzmann equation.

In this work we explore the most natural 3+1 decomposition, one that is not enforced by an artificial external slicing condition. The time slices in the orthogonal comoving coordinates used by Misner and Sharp [4] are attached to the dynamical motion of the matter. As for the basis of the momentum space we first investigate the description in the comoving frame. In another part, we also develop a formulation in the spirit of the tangent ray approach and draw the connections between these two choices. But first of all, we continue with a motivation of the comoving coordinate choice.

The most obvious advantage is already contained in the word *comoving*. A given coordinate interval moves with the matter and adjusts to the location where the matter is. Comoving coordinates ease the description of the internal physical state of fluid elements because numerically difficult advection terms do not enter the hydrodynamics equations as in other coordinate systems. Gravitational collapse in spherical symmetry is a natural application of comoving coordinates. With regard to radiation transport, the comoving frame, which is preferred for the evaluation of the integrals over the angle dependent emission, absorption, and scattering kernels, is collinear with the comoving coordinates such that Lorentz transformations are avoided in the collision term.

On the other hand, the orthogonal comoving coordinates show an important drawback in general relativity. After the formation of a black hole, observers may fall within finite time into the physical singularity at the symmetry center. In addition, a coordinate singularity forms at the Schwarzschild horizon. Simulations of the outer regions cannot proceed because of these singularities in the computational domain.

To circumvent this difficulty, interest focused on singularity avoiding time slicing. For example, simulations of Wilson [12] and Shapiro and Teukolsky [13] were carried out in maximal slicing. The collapse to a black hole can be followed beyond the appearance of trapped surfaces in these coordinates. At late times however, the grid is “sucked down” the black hole, leading to unsatisfactory resolution in the physically interesting region outside the event horizon. Polar slicing shows even stronger singularity avoidance (Bardeen and Piran [14]). The computational domain stays outside the apparent horizon for all times. This time slicing was implemented by Romero et al. [15] in orthogonal coordinates. Schinder et al. [16] found an interesting compromise by introducing nonzero shift vectors to get comoving coordinates in polar slicing. Orthogonality of the coordinates, however, had to be abandoned. The idea of using observer time coordinates suggested by Hernandez and Misner [17] has recently been used by Baumgarte et al. [18] in a singularity avoiding code. Although coordinate and physical singularities are not encountered in singularity avoiding schemes, the coefficients of the metric can increase without limits for late times (Petrich et al. [19]). Alternatively, the computational domain can also be limited to the region outside of the event horizon by the choice of appropriate shift vectors (Liebendörfer and Thielemann [20]). A free choice of time slicing is then reestablished even in the presence of singularities.

One argument against the use of comoving orthogonal coordinates is the difficulty to extend them to multiple spatial dimensions because of grid-entangling. However, one-dimensional simulations offer the opportunity to include general relativity, exact Boltzmann radiation transport, and sophisticated physics for emission, absorption, and scattering at once. While the individual pieces are well represented in the literature, dynamical simulations with all ingredients are very difficult and about to become current state of the art.

Many spherically symmetric simulations of compact objects were based on the comoving orthogonal coordinates of Misner and Sharp [4]. Finite difference schemes were constructed by May and White [3], Van Riper [21], Bruenn [22], and Swesty [23]. An approximate Riemann solver was constructed by Yamada [24]. However, the appearance of the time dependent metric in the general relativistic equations prevented an immediate application of numerical methods developed for conservatively formulated hydrodynamics. A formulation of the dynamics in terms of conservation laws has several benefits: *(i)* Fundamental conservation laws are also valid at discontinuities and allow an accurate numerical solution - e.g., for the propagation of shock waves. This is not the case with arbitrary finite difference approximations. *(ii)* The integration of a conserved quantity over adjacent fluid elements does not depend on the fluxes at the enclosed boundaries. Thus, discretization errors have mainly local influence. This advantage is important for the implicit solution of problems involving scale differences of many orders of magnitude. *(iii)* The integration over the whole computational domain of the single-fluid-element conservation laws leads to the total conservation of fundamental physical quantities, independent of the resolution in the conservative finite difference representation. *(iv)* An overwhelming variety of generalized investigations and numerical methods for the solution of hyperbolic conservation laws already exists (see, e.g., Davis [25] and references therein). In analogy to the work of Romero et al. [15] in polar slicing, Liebendörfer and Thielemann [20] formulated conservative equations for hydrodynamics in the comoving frame. This approach is extended here to include radiation transport. The description (with the omission of neutrino back reaction to the fluid metric) has successfully been used in a general relativistic simulation of stellar core collapse, bounce and postbounce evolution, based on multigroup Boltzmann neutrino transport (Liebendörfer et al. [26,27]).

II. CONSERVATIVE EINSTEIN EQUATIONS

The left-hand side of the Einstein field equation, the Einstein tensor, is determined by the metric. Following Misner and Sharp [4], we select in spherical symmetry

$$ds^2 = -\alpha^2 dt^2 + \left(\frac{r'}{\Gamma}\right)^2 da^2 + r^2 (d\vartheta^2 + \sin^2 \vartheta d\varphi^2), \quad (1)$$

where r is the areal radius and a is a label corresponding to an enclosed rest mass (the prime denotes a derivative with respect to a : $r' = \partial r / \partial a$). The proper time lapse of an observer attached to the motion of rest mass is related to the coordinate time dt by the lapse function α . The angles ϑ and φ describe a two-sphere.

The right-hand side of the Einstein equations is given by the fluid- and radiation stress-energy tensor, T , which, in a comoving orthonormal basis, has the components

$$\begin{aligned} T^{tt} &= \rho(1 + e) \\ T^{ta} &= T^{at} = q \\ T^{aa} &= T^{\vartheta\vartheta} = T^{\varphi\varphi} = p. \end{aligned} \quad (2)$$

The total energy is expressed in terms of the rest mass density, ρ , and the specific energy, e . The isotropic pressure is denoted by p , and radial net energy transport is included by the nondiagonal component q . Note our convention that energy density and pressure contain the contribution from the radiation field as well as the matter.

In Appendix A, we rederive a system of equations equivalent to the Einstein equations in spherical symmetry, closely following the guideline of Misner et al. [28]. The only difference to previous work is the omission of any approximations in the derivation. Moreover, the Appendix gives many useful relations for what follows. The final system of exact equations reads:

$$\frac{\partial r}{\alpha \partial t} = u \quad (3)$$

$$\frac{\partial m}{\alpha \partial t} = -4\pi r^2 (up + \Gamma q) \quad (4)$$

$$\frac{\partial u}{\alpha \partial t} = \frac{\Gamma^2}{\alpha} \frac{\partial \alpha}{\partial r} - \frac{m + 4\pi r^3 p}{r^2} \quad (5)$$

$$r' = \frac{\Gamma}{4\pi r^2 \rho} \quad (6)$$

$$m' = \Gamma(1 + e) + u \frac{q}{\rho} \quad (7)$$

$$\alpha' = -\frac{\alpha}{(1 + e + p/\rho)} \left[\frac{p'}{\rho} + \frac{\partial}{\alpha \partial t} \left(\frac{1}{4\pi r^2 \rho} \frac{q}{\rho} \right) \right]. \quad (8)$$

The velocity u is equivalent to the r -component of the fluid four-velocity as observed from a frame at constant areal radius (May and White [3]). In the special relativistic limit, $\Gamma = \sqrt{1 + u^2 - 2m/r}$ becomes the Lorentz factor corresponding to the boost between the inertial and the comoving observers. The gravitational mass m is the total energy enclosed in the sphere at rest mass a . Its change is determined by the surface work, involving pressure p and velocity u , and the boundary luminosity $L = 4\pi r^2 q$.

We will now show that Eqs. (3-8) can be written in conservative form. Optimal for a numerical implementation is a formulation that contains quantities that are either conserved or local. Since the rest mass as independent variable is trivially conserved, we look out for the conservation of volume, energy and momentum. Eq. (6) leads to the definition of the integral

$$V \equiv \frac{4\pi}{3} r^3 = \int_0^a \frac{\Gamma}{\rho} da \quad (9)$$

as volume and Eq. (7) suggests the definition of the integral

$$m = \int_0^a \left[\Gamma(1 + e) + \frac{uq}{\rho} \right] da \quad (10)$$

as gravitational mass, equivalent to total enclosed energy. For the radial momentum we choose $u(1 + e) + \Gamma q/\rho$ and show later that it indeed leads to a conservation equation. Borrowing the notation from Romero et al. [15], we rename the candidates for conserved quantities and define specific volume, specific energy and specific momentum as

$$\frac{1}{D} = \frac{\Gamma}{\rho} \quad (11)$$

$$\tau = \Gamma e + \frac{2}{\Gamma + 1} \left(\frac{1}{2} u^2 - \frac{m}{r} \right) + \frac{uq}{\rho} \quad (12)$$

$$S = u(1 + e) + \Gamma \frac{q}{\rho}. \quad (13)$$

In the nonrelativistic limit, we retrieve with $\alpha = \Gamma = 1$ the familiar specific volume $D = 1/\rho$, the sum of the specific internal, kinetic, and gravitational energy $\tau = e + u^2/2 - m/r$, and the specific radial momentum $S = u$. With the help of Eqs. (3) and (6), the time derivative of $1/D$ leads to the continuity equation (15). An energy equation is obtained by taking the time derivative of specific total energy $1 + \tau$ and substituting Eq. (4):

$$\begin{aligned} \frac{\partial}{\alpha \partial t} \left[\Gamma (1 + e) + u \frac{q}{\rho} \right] &= \frac{\partial}{\alpha \partial t} (m') = \left(\frac{\partial m}{\alpha \partial t} \right)' + \frac{\alpha'}{\alpha} \frac{\partial m}{\alpha \partial t} \\ &= -\frac{1}{\alpha} \frac{\partial}{\partial a} [4\pi r^2 \alpha (up + \Gamma q)]. \end{aligned} \quad (14)$$

This immediately leads to the conservation equation (16) for total energy. A tedious but straightforward calculation based on Eqs. (14), (A3-A10), and (A13) finally leads to the momentum equation (17) that completes the set of conservation equations:

$$\frac{\partial}{\partial t} \left[\frac{1}{D} \right] = \frac{\partial}{\partial a} [4\pi r^2 \alpha u] \quad (15)$$

$$\frac{\partial \tau}{\partial t} = -\frac{\partial}{\partial a} [4\pi r^2 \alpha (up + \Gamma q)] \quad (16)$$

$$\begin{aligned} \frac{\partial S}{\partial t} &= -\frac{\partial}{\partial a} [4\pi r^2 \alpha (\Gamma p + uq)] \\ &\quad - \frac{\alpha}{r} \left[\left(1 + e + \frac{3p}{\rho} \right) \frac{m}{r} + \frac{8\pi r^2}{\rho} (\rho (1 + e) p - q^2) - \frac{2p}{\rho} \right] \end{aligned} \quad (17)$$

$$\frac{\partial V}{\partial a} = \frac{1}{D} \quad (18)$$

$$\frac{\partial m}{\partial a} = 1 + \tau \quad (19)$$

$$\frac{\partial}{\partial t} \left[\frac{1}{4\pi r^2 \rho} \frac{q}{\rho} \right] = -(1 + e) \frac{\partial \alpha}{\partial a} - \frac{1}{\rho} \frac{\partial}{\partial a} [\alpha p]. \quad (20)$$

The time derivative in the last equation usually is small, so that this equation rather acts as a constraint on the lapse function α . This equation is derived in Appendix A from the space-space component of the four divergence of the stress-energy tensor. The constraints (18) and (19) explain themselves in the analogy to the Newtonian limit, where the first becomes the definition of the rest mass density and the second the Poisson equation for the gravitational potential. This set of conservative equations is fundamental for the following discussion. We will first point out their relation to the general relativistic jump conditions at a shock front and then derive a consistent incorporation of artificial viscosity in Eqs. (15-20). In a second and third part, we investigate the general relativistic Boltzmann equation with two different natural descriptions of the momentum phase space for the radiation particles. Both formulations will show interesting relations to the above-mentioned conservation laws.

III. SHOCK WAVES AND ARTIFICIAL VISCOSITY

The jump conditions across a shock front reflect the governing conservation laws and can directly be determined from the latter: In an isolated shock wave, we assume a stable physical state at the left-hand side (subscript L) and at the right-hand side (index R). The direction of shock propagation will not play a role in this derivation. A conservation law can then be integrated over an infinitesimal range around the shock front, containing rest mass Δa_L on the left-hand side and Δa_R on the right-hand side of the shock at position a_s . For example, the continuity equation (15) leads to

$$\frac{d}{dt} \left(\frac{\Delta a_L}{D_L} + \frac{\Delta a_R}{D_R} \right) = 4\pi r_R^2 \alpha_R u_R - 4\pi r_L^2 \alpha_L u_L.$$

As the shock moves with $da_s/dt = da_L/dt = -da_R/dt$ through rest mass the first jump condition reads

$$\left[\frac{da_s}{dt} \frac{1}{D} + 4\pi r^2 u \alpha \right] = 0. \quad (21)$$

The bracket denotes the difference of the enclosed expression evaluated on both sides of the shock. With an analogous argument applied to the energy conservation equation, the general relativistic jump condition in spherical symmetry with energy transport included reads:

$$\left[\frac{da_s}{dt} \tau - 4\pi r^2 \alpha (up + \Gamma q) \right] = 0. \quad (22)$$

At this point one might guess the correct jump conditions for the momentum conservation from Eq. (17) - in spite of the complications introduced by the nonvanishing source term. A rigorous derivation along the lines given by May and White [3] indeed leads to

$$\left[\frac{da_s}{dt} S - 4\pi r^2 \alpha (\Gamma p + uq) \right] = 0. \quad (23)$$

Of course, with $q = 0$, one immediately recovers the jump conditions for pure hydrodynamics found in Ref. [3].

Although ideal hydrodynamics allows discontinuities in the physical state across a shock front, there are many numerical schemes that cannot handle infinite gradients in the velocity, density, or temperature profiles. A well known solution to this problem is the introduction of an artificial viscosity. The artificial viscosity limits the gradient of the profiles and broadens the shock, which then becomes spread over several grid zones. With a conservative formulation, the dynamics of these asymptotically incompressible zones becomes determined by conservation of volume, energy, and momentum flux across them. This guarantees the conservation of these quantities over the whole shock as required by the jump conditions. Older schemes use artificial viscosity in a form proposed by Von Neumann and Richtmyer [29].

In spherical symmetry, the formulation of artificial viscosity has to be chosen with care in order to avoid systematic artificial heating during homologous compression. An excellent solution has been found by Tscharnuter and Winkler [30] for nonrelativistic hydrodynamics. We extend this approach to general relativistic hydrodynamics and define the viscous tensor

$$\begin{aligned} Q_{\alpha\beta} &= \Delta l^2 \rho u^\mu_{;\mu} \left[\varepsilon_{\alpha\beta} - \frac{1}{3} u^\mu_{;\mu} P_{\alpha\beta} \right] \quad \text{if } u^\mu_{;\mu} < 0, \\ &= 0 \quad \text{otherwise;} \\ \varepsilon_{\alpha\beta} &= \frac{1}{2} \left(u_{\alpha;\mu} P^\mu_\beta + u_{\beta;\mu} P^\mu_\alpha \right), \end{aligned}$$

where $P_{\alpha\beta} = u_\alpha u_\beta + g_{\alpha\beta}$ is the projection operator onto the three-space orthogonal to the fluid four-velocity u^α . The artificial viscosity is based on physical viscosity and is chosen according to the standard shear viscosity (Misner et al. [28], exercise (22.6)). It is weighted with a variable coefficient that reflects the local density and compression. The length-scale Δl sets the order of magnitude of the desired shock width.

First we note that the artificial viscosity tensor is traceless, and that in our comoving coordinates its nonvanishing components are:

$$\begin{aligned} Q^a_a &= -2Q^\vartheta_\vartheta = -2Q^\varphi_\varphi = Q, \\ Q &= \Delta l^2 \rho \text{div}(u) \left[\frac{\partial u}{\partial r} - \frac{1}{3} \text{div}(u) \right] \quad \text{if } \text{div}(u) < 0, \\ &= 0 \quad \text{otherwise;} \\ \text{div}(u) &= \frac{\partial}{\partial V} (4\pi r^2 u). \end{aligned} \quad (24)$$

This viscous tensor is included in the derivation of the Einstein equations in spherical symmetry in Appendix A. We can check the behaviour of viscous heating in the energy equation (A12):

$$\begin{aligned} \frac{\partial e}{\alpha \partial t} &= \left(\frac{u}{r} - \frac{\partial u}{\partial r} \right) \frac{Q}{\rho} \\ &= -\frac{3}{2} \left[\frac{\partial u}{\partial r} - \frac{1}{3} \text{div}(u) \right] \frac{Q}{r}. \end{aligned}$$

The first expression shows that the viscous heating vanishes in the case of homologous compression ($u/r = \partial u / \partial r$). This is due to the nonisotropy of the viscous pressure: With homologous compression, the work done on a fluid element

by radial compression against Q_a^a would heat the fluid element. But the homologous compression comes together with a simultaneous compression in ϑ and φ direction. By choosing the viscous pressure components to have negative sign in these directions, the net heat production vanishes. The second expression together with Eq. (24) shows that viscous heating always has positive sign. In a conservative formulation, the viscosity affects the equation for the total energy evolution (16), the momentum evolution (17), and the constraint for the lapse function (20). These equations then become

$$\begin{aligned}\frac{\partial \tau}{\partial t} &= -\frac{\partial}{\partial a} [4\pi r^2 \alpha (u(p+Q) + \Gamma q)] \\ \frac{\partial S}{\partial t} &= -\frac{\partial}{\partial a} [4\pi r^2 \alpha (\Gamma(p+Q) + uq)] \\ &\quad - \frac{\alpha}{r} \left[\left(1 + e + \frac{3(p-Q)}{\rho} \right) \frac{m}{r} + \frac{8\pi r^2}{\rho} (\rho(1+e)(p+Q) - q^2) - \frac{2p}{\rho} + \frac{Q}{\rho} \right] \\ \frac{\partial}{\partial t} \left[\frac{1}{4\pi r^2 \rho} \frac{q}{\rho} \right] &= -(1+e) \frac{\partial \alpha}{\partial a} - \frac{1}{\rho} \frac{\partial}{\partial a} [\alpha p] - \frac{1}{V\rho} \frac{\partial}{\partial a} [V\alpha Q].\end{aligned}$$

IV. BOLTZMANN RADIATION TRANSPORT

A. General Relativistic View

The general relativistic Boltzmann equation in comoving coordinates was derived by Lindquist [5]. Radiation that is not necessarily in equilibrium with the matter is described by a distribution function f ,

$$dN = f(x, p) (-p_\mu u^\mu) dV dP.$$

The volume element dV is crossed at four-location x by dN radiation particle world lines with four-momenta p in the range dP while the observer moves with four-velocity u^μ . Lindquist [5] shows that this definition makes the distribution function Lorentz invariant. We measure the particle four-momentum in a comoving orthonormal frame (A1), with the components

$$p^a = p \cos \theta, \quad p^\vartheta = p \sin \theta \cos \phi, \quad p^\varphi = p \sin \theta \sin \phi. \quad (25)$$

The absolute value, p , of the three-momentum can be determined from the scalar particle energy $E = -p_\mu u^\mu = p^t = \sqrt{p^2 + m^2}$ and the rest mass. (the mass of radiation particles is assumed to be zero in this paper). The direction of the particle three momentum is specified by the angle cosine $\mu = \cos \theta$ to the radial direction. In spherical symmetry, the distribution function does not depend on the azimuth angle ϕ . Thus, the particle distribution function depends on four arguments:

$$dN = f(t, a, \mu, E) E^2 dE d\mu dV.$$

The Boltzmann equation for metric (1) then reads (Lindquist [5], Eq. (3.7))

$$\begin{aligned}\frac{1}{\alpha} \frac{\partial f}{\partial t} + \mu \frac{r'}{\Gamma} \frac{\partial f}{\partial a} + \frac{\Gamma}{r} (1 - \mu^2) \frac{\partial f}{\partial \mu} \\ - \left[\Gamma \frac{\partial \Phi}{\partial r} + \mu \frac{1}{\alpha} \frac{\partial \Lambda}{\partial t} - \mu \frac{u}{r} \right] (1 - \mu^2) \frac{\partial f}{\partial \mu} \\ - \left[\mu \Gamma \frac{\partial \Phi}{\partial r} + \mu^2 \frac{1}{\alpha} \frac{\partial \Lambda}{\partial t} + (1 - \mu^2) \frac{u}{r} \right] E \frac{\partial f}{\partial E} = j + \chi f,\end{aligned} \quad (26)$$

with $\Lambda = \log(r'/\Gamma)$. The right-hand side of the equation is the collision term that describes changes in the particle distribution function due to interactions with matter. It is represented here by an emissivity j and an absorptivity χ . Since the independent space coordinate is enclosed rest mass, it is convenient to introduce the specific distribution function $F(t, a, \mu, E) = f(t, a, \mu, E)/\rho$. With the help of the relations in Appendix A, the Boltzmann equation (26) can be rewritten in conservative form (Mezzacappa and Matzner [8], Yamada et al. [31])

$$\begin{aligned}
& \frac{\partial F}{\alpha \partial t} + \frac{\mu}{\alpha} \frac{\partial}{\partial a} [4\pi r^2 \alpha \rho F] + \Gamma \left(\frac{1}{r} - \frac{1}{\alpha} \frac{\partial \alpha}{\partial r} \right) \frac{\partial}{\partial \mu} [(1 - \mu^2) F] \\
& + \left(\frac{\partial \ln \rho}{\alpha \partial t} + \frac{3u}{r} \right) \frac{\partial}{\partial \mu} [\mu (1 - \mu^2) F] \\
& + \left(\mu^2 \left(\frac{\partial \ln \rho}{\alpha \partial t} + \frac{3u}{r} \right) - \frac{u}{r} - \mu \Gamma \frac{1}{\alpha} \frac{\partial \alpha}{\partial r} \right) \frac{1}{E^2} \frac{\partial}{\partial E} [E^3 F] = \frac{j}{\rho} + \chi F.
\end{aligned} \tag{27}$$

The first term is the temporal change of the particle distribution function. The second term counts the particles that are propagating into or out of an infinitesimal mass shell. The term $\frac{\Gamma}{r} \frac{\partial}{\partial \mu} [(1 - \mu^2) F]$ corrects for the change in the local propagation angle μ when the particle moves to another radius. The curved particle trajectory in general relativity is accounted for by the term $\frac{\Gamma}{\alpha} \frac{\partial \alpha}{\partial r} \frac{\partial}{\partial \mu} [(1 - \mu^2) F]$ that is proportional to the gradient of the gravitational potential $\frac{1}{\alpha} \frac{\partial \alpha}{\partial r} = \frac{\partial \Phi}{\partial r}$. The term on the second line accounts for angular aberration: When the comoving observers change radius, the corresponding relabeling of particle propagation angles also adds a correction to the in/out flow balance for the particle distribution function taken at constant μ . The same is true for the frequency shift described in the third line: The first two terms in the parenthesis account for the Doppler shift caused by comoving observers. The relativistic term $\mu \Gamma \frac{\partial \Phi}{\partial r}$ describes the red- or blue shift in the particle energy that applies when the particles have a velocity component in the radial direction ($\mu \neq 0$) and therefore change their position in the gravitational potential.

The integration of the Boltzmann equation over momentum space, spanned by the propagation angle and particle energy, is supposed to reproduce the conservation laws for particle number and energy as stated in Eqs. (15) and (16). We define H^N and G^N to represent the zeroth and first μ -moment of the distribution function

$$\begin{aligned}
H^N &= \int_{-1}^1 \int_0^\infty F E^2 dE d\mu, \\
G^N &= \int_{-1}^1 \int_0^\infty F E^2 dE \mu d\mu.
\end{aligned}$$

Integration of Eq. (27) over momentum space gives the following evolution equation in terms of these moments:

$$\frac{\partial H^N}{\partial t} + \frac{\partial}{\partial a} [4\pi r^2 \alpha \rho G^N] - \alpha \int \frac{j}{\rho} E^2 dE d\mu + \alpha \int \chi F E^2 dE d\mu = 0. \tag{28}$$

The aberration and frequency shift terms do not contribute because $(1 - \mu^2)$ vanishes at $\mu = \pm 1$ and $E^3 F$ is zero for $E = 0$ as well as for $E = \infty$. The nature of Eq. (28) is a continuity equation analogous to Eq. (15), extended by source and absorption terms for the radiation particles. One more integration over the rest mass a from the center of the star to its surface finally gives an equation for the total radiation particle number variation as a function of integrated emission and absorption.

Slightly less straightforward is the reproduction of the energy conservation. Defining the energy moments as

$$\begin{aligned}
H^E &= \int F E^3 dE d\mu \\
G^E &= \int F E^3 dE \mu d\mu = q/\rho \\
P^E &= \int F E^3 dE \mu^2 d\mu,
\end{aligned}$$

the corresponding integration of the Boltzmann equation (27) first leads to the radiation energy equation and the radiation momentum equation

$$\begin{aligned}
& \frac{\partial H^E}{\partial t} + \frac{\partial}{\partial a} [4\pi r^2 \alpha \rho G^E] + \frac{\alpha u}{r} (H^E - P^E) - \left(\frac{\partial \ln \rho}{\partial t} + \frac{2\alpha u}{r} \right) P^E \\
& + \Gamma \frac{\partial \alpha}{\partial r} G^E - \alpha \int \frac{j}{\rho} E^3 dE d\mu + \alpha \int \chi F E^3 dE d\mu = 0, \\
& \frac{\partial G^E}{\partial t} + \frac{\partial}{\partial a} [4\pi r^2 \alpha \rho P^E] - \left(\frac{\alpha \Gamma}{r} - \Gamma \frac{\partial \alpha}{\partial r} \right) (H^E - P^E) - \left(\frac{\partial \ln \rho}{\partial t} + \frac{2\alpha u}{r} \right) G^E \\
& + \Gamma \frac{\partial \alpha}{\partial r} P^E + \alpha \int \chi F E^3 dE \mu d\mu = 0.
\end{aligned}$$

Having Eq. (14) in mind, we construct, in analogy to $\Gamma(1+e) + uq/\rho$, the specific radiation energy $\Gamma H^E + uG^E$ and investigate its time derivative. After a fair amount of algebra and the use of relations from Appendix A, most of the contributions cancel and one is left with the expected result

$$0 = \frac{\partial}{\partial t} (\Gamma H^E + uG^E) + \frac{\partial}{\partial a} [4\pi r^2 \alpha \rho (uP^E + \Gamma G^E)] \\ - \alpha \Gamma \int \frac{j}{\rho} E^3 dE d\mu + \alpha \Gamma \int \chi F E^3 dE d\mu + \alpha u \int \chi F E^3 dE \mu d\mu. \quad (29)$$

The time derivative contains the conserved energy, the second term describes the change of energy in the fluid element due to surface work by the radiation pressure, P^E , and in/out flow of radiation at luminosity $L = 4\pi r^2 \rho G^E$. Note that $(uP^E + \Gamma G^E)$, being the first μ -moment of $(\Gamma H^E + uG^E)$, indeed is the flux of the conserved quantity at the zone boundary. The same feature is realized in the radiation momentum equation (17): $\partial/\partial t (uH^E + \Gamma G^E) = -\partial/\partial a [4\pi r^2 \alpha \rho (\Gamma P^E + uG^E)] + \dots$. It underlines the natural appearance of this formulation. The second line in Eq. (29) describes the energy exchange with matter by emission, absorption, and work due to particle stress. Thus, by splitting the internal energy $e = \tilde{e} + H^E$ and pressure $p = \tilde{p} + P^E$ into contributions from the matter stress-energy tensor and the radiation stress-energy tensor, the exact energy conservation equation can finally be written concisely as

$$\frac{\partial}{\partial t} \left[\Gamma \tilde{e} + \frac{2}{\Gamma+1} \left(\frac{1}{2} u^2 - \frac{m}{r} \right) + \Gamma H^E + uG^E \right] \\ + \frac{\partial}{\partial a} [4\pi r^2 \alpha u \tilde{p} + 4\pi r^2 \alpha \rho (uP^E + \Gamma G^E)] = 0. \quad (30)$$

B. The Order v/c Limit

The $O(v/c)$ Boltzmann equation was derived by Castor [32]. One can also obtain it by directly dropping the higher order terms from Eq. (27), i.e., by setting $\alpha = \Gamma = 1 = \text{const.}$ and replacing u by the nonrelativistic velocity v :

$$\frac{\partial F}{\partial t} + \mu \frac{\partial}{\partial a} [4\pi r^2 \rho F] + \frac{1}{r} \frac{\partial}{\partial \mu} [(1 - \mu^2) F] \\ + \left(\frac{\partial \ln \rho}{\partial t} + \frac{3v}{r} \right) \frac{\partial}{\partial \mu} [\mu (1 - \mu^2) F] \\ + \left(\mu^2 \left(\frac{\partial \ln \rho}{\partial t} + \frac{3v}{r} \right) - \frac{v}{r} \right) \frac{1}{E^2} \frac{\partial}{\partial E} [E^3 F] = \frac{j}{\rho} + \chi F. \quad (31)$$

The conservation properties of the $O(v/c)$ Boltzmann equation become apparent in terms of the moments that are directly derived from Eq. (31):

$$\frac{\partial H^E}{\partial t} + \frac{\partial}{\partial a} [4\pi r^2 \rho G^E] + \frac{v}{r} (H^E - P^E) - \left(\frac{\partial \ln \rho}{\partial t} + \frac{2v}{r} \right) P^E \\ - \int \frac{j}{\rho} E^3 dE d\mu + \int \chi F E^3 dE d\mu = 0, \\ \frac{\partial G^E}{\partial t} + \frac{\partial}{\partial a} [4\pi r^2 \rho P^E] - \frac{1}{r} (H^E - P^E) - \left(\frac{\partial \ln \rho}{\partial t} + \frac{2v}{r} \right) G^E \\ + \int \chi F E^3 dE \mu d\mu = 0.$$

When we construct the conserved specific radiation energy $(\Gamma H^E + uG^E)$ in the $O(v/c)$ limit, we first keep all terms that arise from the $O(v/c)$ Boltzmann equation and obtain

$$0 = \frac{\partial}{\partial t} (H^E + vG^E) + \frac{\partial}{\partial a} [4\pi r^2 \rho (vP^E + G^E)] \\ - \int \frac{j}{\rho} E^3 dE d\mu + \int \chi F E^3 dE d\mu + v \int \chi F E^3 dE \mu d\mu \\ - \frac{1}{4\pi r^2 \rho} \frac{\partial}{\partial t} (4\pi r^2 \rho v) G^E. \quad (32)$$

Note that the first order terms $\frac{v}{r}(H^E - P^E) - \left(\frac{\partial \ln \rho}{\partial t} + \frac{2v}{r}\right)P^E$ arising from frequency shift and angular aberration in the radiation energy equation cancel with the zeroth order terms in the radiation momentum equation after the latter are multiplied by the fluid velocity. In a numerical implementation, the finite differencing of the Boltzmann equation has to guarantee the same cancellation in order to achieve $O(v/c)$ energy conservation. The nonconservative term found in the third line in Eq. (32) is of second order in v/c .

V. APPROXIMATE TANGENT RAY DESCRIPTION

A. General Relativistic View

The description of the particle four-momenta in the comoving frame eases the numerical implementation of the collision term. The angle dependent integrals over reaction cross sections of particles with matter are readily evaluated if the target is at rest in the comoving frame. On the other hand, the left-hand side of the Boltzmann equation (27) has to take care about the Lorentz transformation between adjacent comoving observers in the description of the changes in the particle distribution function due to propagation. An adequate discretization of the numerous correction terms with partial derivatives is, in spherical symmetry, a resolved challenge (Mezzacappa and Bruenn [33], Liebendörfer [26]). However, this does not prevent consideration of other interesting options. Here we would like to look into a description of the momentum phase space in variables that remain constant along a propagation path in the absence of interactions. Particles change their coordinates in the momentum phase space only due to collisions, evaluated on the right-hand side of the Boltzmann equation. The left-hand side does not require corrections involving partial derivatives with respect to the coordinates of the momentum phase space. The drawback is given by the unavoidable need of transformations to relate the cross sections in the fluid rest frame to the distribution function in the chosen momentum phase space description. This idea was investigated in polar slicing and the radial gauge by Schinder and Bludman [11]. We explore a similar ansatz in orthogonal comoving coordinates in this section.

A geodesic in a static spherically symmetric space-time can uniquely be described by an impact parameter b and a particle energy ε at infinity. Let us start with a Schwarzschild metric

$$ds^2 = -\Gamma_S^2 dt_S^2 + \Gamma_S^{-2} dr^2 + r^2 (d\vartheta + \sin^2 \vartheta d\varphi^2), \quad (33)$$

where we add a subscript S to quantities that could be confused with corresponding quantities in the comoving frame (e.g., $\Gamma_S = \sqrt{1 - 2m/r}$). The trajectory of free propagation in the plane of constant φ with energy E_S is related to the impact parameter b and the energy at infinity ε as (Misner et al. [28], Eq. (25.55) and (25.18/19))

$$\left(\frac{1}{r} \frac{dr}{d\vartheta}\right)^2 + 1 - \frac{2m}{r} = \frac{r^2}{b^2} \quad (34)$$

$$\Gamma_S E_S = \varepsilon. \quad (35)$$

The relation between the particle propagation in the r - and ϑ -directions is given by the angle θ the particle trajectory makes with the outward radial direction

$$\tan \theta = \frac{\sqrt{1 - \mu_S^2}}{\mu_S} = \frac{r d\vartheta}{\Gamma_S^{-1} dr}. \quad (36)$$

Eqs. (34) and (36) are solved for the relation between the particle angle, μ_S , and the impact parameter,

$$b = \frac{r}{\Gamma_S} \sqrt{1 - \mu_S^2}. \quad (37)$$

Our next step is to transform μ_S and E_S to the particle angle, μ , and energy, E , measured by the comoving observers. From the metrics (1) and (33) one obtains the relations

$$\begin{aligned} \frac{dt_S}{dt} &= \frac{\alpha \Gamma}{\Gamma_S^2} \\ \frac{\partial t_S}{\partial a} &= \frac{r'}{\Gamma} \frac{u}{\Gamma_S^2} \end{aligned} \quad (38)$$

for the Lorentz transformation between the Schwarzschild- and comoving-coordinate time. The Lorentz transformation also links the particle four-momenta in these two coordinate bases,

$$\begin{aligned}
p_S^\mu &= \left(\Gamma_S^{-1}, \Gamma_S \mu_S, \frac{1}{r} \sqrt{1 - \mu_S^2}, 0 \right) E_S \\
p^\mu &= \left(\alpha^{-1}, \frac{\Gamma}{r'} \mu, \frac{1}{r} \sqrt{1 - \mu^2}, 0 \right) E.
\end{aligned} \tag{39}$$

From the application of transformation (38) to the four-momenta in (39), we extract the transformation of the particle propagation angle and energy:

$$\begin{aligned}
\frac{1 - \mu_S^2}{\Gamma_S^2} &= \frac{1 - \mu^2}{(\Gamma + u\mu)^2} \\
\Gamma_S E_S &= (\Gamma + u\mu) E.
\end{aligned}$$

Finally, Eqs. (37) and (35) become in terms of comoving frame variables:

$$b = r \frac{\sqrt{1 - \mu^2}}{\Gamma + u\mu} \tag{40}$$

$$\varepsilon = (\Gamma + u\mu) E. \tag{41}$$

These useful relations link the local particle angles and energies measured by comoving observers at different radii and with different velocities to the particle impact parameter and energy at infinity. Additionally, these relations provide insight into the nature of the energy conservation equation (29). The conserved quantity under the time derivative

$$\Gamma H^E + u G^E = \int (\Gamma + u\mu) F E^3 dE d\mu = \int \varepsilon F E^2 dE d\mu$$

is the total radiation energy *at infinity* expressed in terms of the comoving frame radiation energy and momentum. This also makes clear the following important point: The locally observed radiation quantities must be transformed to a common observation point (e.g., infinity) before they can be integrated to define a conserved quantity.

Let us recall that the particle trajectories along constant impact parameter, b , and constant energy at infinity, ε , were derived in static vacuum Schwarzschild space-time surrounding a gravitational mass m . Nevertheless, we can span the momentum phase space in the dynamical Boltzmann equation in comoving coordinates, Eq. (27), with the coordinates (b, ε) ¹ instead of the comoving frame four-momenta (μ, E) . If the static trajectories are a good approximation to the trajectories on a dynamical background, we expect that the momentum state of free particle propagation, measured in (b, ε) , barely changes between adjacent comoving observers. This would imply that the terms involving partial derivatives with respect to b and ε become small compared to the time derivative of the distribution function and the in/out flow term on the left-hand side of the exact Boltzmann equation.

We replace the partial derivatives in the Boltzmann equation by directional derivatives along the static particle trajectory: We add a bar to the distribution function $\bar{f}(t, a, b, \varepsilon) = f(t, a, \mu, E)$ in order to indicate that the partial derivatives are taken at constant impact parameter b and energy at infinity ε and make the following ansatz:

$$C_1 \frac{\partial \bar{f}}{\partial t} + C_2 \frac{\partial \bar{f}}{\partial a} + C_3 \frac{\partial \bar{f}}{\partial b} + C_4 \frac{\partial \bar{f}}{\partial \varepsilon} = j + \chi \bar{f}. \tag{42}$$

The coefficients C_i are calculated by comparing the Boltzmann equation (26) to (42). The relations (40) and (41) are used in a straightforward but lengthy replacement of the partial derivatives in Eq. (26) by derivatives with respect to impact parameter and energy at infinity. The Boltzmann equation written in terms of these directional derivatives reads

$$\begin{aligned}
&\frac{1}{\alpha} \frac{\partial \bar{f}}{\partial t} + \mu \frac{\Gamma}{r'} \frac{\partial \bar{f}}{\partial a} + \frac{4\pi r}{\Gamma + u\mu} \left(\mu \rho \left(1 + e + \frac{p}{\rho} \right) - (1 + \mu^2) q \right) \\
&\times \left(-\varepsilon \frac{\partial \bar{f}}{\partial \varepsilon} + b \frac{\partial \bar{f}}{\partial b} \right) = j + \chi \bar{f}.
\end{aligned} \tag{43}$$

¹Strictly speaking, the pair (b, ε) only specifies a trajectory. The propagation direction on this trajectory can, for example, be addressed with the convention that a negative impact parameter denotes a propagation with decreasing radius and a positive impact parameter a propagation with increasing radius on the same trajectory.

Its left-hand side involves the first two usual propagation terms that relate the change in the distribution function to particle in/out flow at the boundaries of a mass element. The third term arises owing to the drift of the particle location in the phase space from constant (b, ε) in front of a non-vacuum background. This general relativistic term is of order $(G\rho r^2/c^2)$ and vanishes in the vacuum. It is interesting to see in the static limit, that this term mainly arises from the background presence of matter and not from its dynamical motion. We might then neglect this $O(G\rho r^2/c^2)$ term in the exact equation (43) and find to the very simple approximative general relativistic Boltzmann equation in spherical symmetry:

$$\frac{1}{\alpha} \frac{\partial \bar{f}}{\partial t} + \mu \frac{\Gamma}{r'} \frac{\partial \bar{f}}{\partial a} = j + \chi \bar{f}. \quad (44)$$

This approximation is excellent in the proximity of compact objects, it includes full gravitational redshift. It might however fail within regions of extremely high energy density. Moreover, we show in the next section that its $O(v/c)$ expansion is identical with the full $O(v/c)$ Boltzmann equation (31). Eq. (44) therefore also incorporates angular aberration and Doppler shift between the comoving coordinate frames. Its intuitive form simply reflects Lindquist's general starting point [5] (2.12/22), where the Boltzmann equation is written in terms of directional derivatives of the distribution function along the phase flow.

B. The Order v/c Limit

First, we multiply the left hand side of the $O(v/c)$ Boltzmann equation (31) by the density, ρ , express it in terms of the neutrino distribution function $f = \rho F$, and take the prefactors out of the derivatives to obtain the formulation presented by Castor [32]. With the continuity equation

$$-\frac{\partial \ln \rho}{\partial t} = \frac{v'}{r'} + \frac{2v}{r} \quad (45)$$

we then find

$$LHS = \frac{\partial f}{\partial t} + \frac{\mu}{r'} \frac{\partial f}{\partial a} + (1 - \mu^2) \left[\mu \left(\frac{v}{r} - \frac{v'}{r'} \right) + \frac{1}{r} \right] \frac{\partial f}{\partial \mu} - \left[\mu^2 \frac{v'}{r'} + (1 - \mu^2) \frac{v}{r} \right] E \frac{\partial f}{\partial E}. \quad (46)$$

On the other hand, we expand the approximate general relativistic Boltzmann equation (44) to order v/c and compare to Eq. (46). Before the expansion, however, we have to transform the directional derivatives back to partial derivatives with respect to comoving frame energy, E , and angle, μ . In terms of an unspecified parameter λ , Eqs. (40) and (41) lead to the derivatives of the comoving frame propagation angle and particle energy, taken at constant impact parameter and energy at infinity:

$$\begin{aligned} \left(\frac{\partial \mu}{\partial \lambda} \right)_{b, \varepsilon} &= (1 - \mu^2) \frac{1 + v\mu}{v + \mu} \left(\frac{1}{r} \frac{\partial r}{\partial \lambda} - \frac{\mu}{1 + v\mu} \frac{\partial v}{\partial \lambda} \right) \\ \left(\frac{\partial E}{\partial \lambda} \right)_{b, \varepsilon} &= -\frac{E}{v + \mu} \left((1 - \mu^2) \frac{v}{r} \frac{\partial r}{\partial \lambda} + \mu^2 \frac{\partial v}{\partial \lambda} \right). \end{aligned} \quad (47)$$

The evaluation of the partial derivatives on the left hand side of Eq. (44) is straightforward with the relations (47). We obtain

$$\begin{aligned} LHS &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mu} \frac{\partial \mu}{\partial t} + \frac{\partial f}{\partial E} \frac{\partial E}{\partial t} + \frac{\mu}{r'} \frac{\partial f}{\partial a} + \frac{\mu}{r'} \frac{\partial f}{\partial \mu} \frac{\partial \mu}{\partial a} + \frac{\mu}{r'} \frac{\partial f}{\partial E} \frac{\partial E}{\partial a} \\ &= \frac{\partial f}{\partial t} + \frac{\mu}{r'} \frac{\partial f}{\partial a} + (1 - \mu^2) \left[(1 + v\mu) \frac{1}{r} - \frac{\mu}{1 + v/\mu} \frac{v'}{r'} \right] \frac{\partial f}{\partial \mu} \\ &\quad - \left[(1 - \mu^2) \frac{v}{r} + \frac{\mu^2}{1 + v/\mu} \frac{v'}{r'} \right] E \frac{\partial f}{\partial E}. \end{aligned} \quad (48)$$

Following Bruenn [22], we have already dropped the terms involving the matter acceleration $\partial v/\partial t$ in the last step. If we further neglect the order v/c and higher order terms, Eq. (48) reduces exactly to Eq. (46). We therefore conclude that the approximate general relativistic Boltzmann equation (44) exactly includes and, moreover, extends the $O(v/c)$ Boltzmann equation in the presence of strong gravitational fields.

We provide the exact Einstein equations for radiation hydrodynamics in spherical symmetry, formulated in the comoving orthogonal coordinates introduced by Misner and Sharp [4], [34]. We show that these equations can be written in a concise and strictly conservative form. The general relativistic jump conditions at shock fronts are derived under inclusion of the radiation energy - and radiation momentum flux.

The conservative formulation is ideally suited for the application of numerous numerical schemes specifically designed for the solution of partial differential equations in this form: for example, adaptive grid techniques (Winkler et al. [35]), Riemann solvers (Godunov [36], Davis [25]), and high-resolution shock capturing schemes (Romero et al. [15], Marti and Müller [37]). Some of these schemes require artificial viscosity for a numerical representation of shock waves. We have generalized the tensor artificial viscosity ansatz of Tscharnuter and Winkler [30] to general relativity. As in the nonrelativistic case, viscous heating is shown to always have positive sign and to vanish during homologous collapse.

In the second part of this paper, we focus on the general relativistic Boltzmann equation in our chosen coordinates. We confirm that, as expected, the appropriate moments of the particle distribution function, evolved according to the Boltzmann transport equation, obey the conservation laws found in the first part of the paper. In the $O(v/c)$ limit, we identify leading terms that require careful discretization in a finite difference representation of the Boltzmann equation. Angular aberration and frequency shift corrections arise in the propagation part of the Boltzmann equation since the time and space derivative of the particle distribution function is taken at constant comoving frame four-momenta in the particle momentum phase space.

In the last part, we try to recover the simple nature of the left-hand side of the transport equation, understanding it as a directional derivative along the phase flow of the particles. The approximation of the phase flow by geodesics in vacuum Schwarzschild space-time leads first to the exact general relativistic Boltzmann equation with the momentum phase space now described in terms of particle impact parameter and energy at infinity. The correction terms to the partial derivatives with respect to time and space along vacuum geodesics are of order $(G\rho r^2/c^2)$ in this representation. They are only due to the background matter distribution neglected in the determination of the phase flow. We show that neglecting these terms leads to a very intuitive Boltzmann equation that is exact in the $O(v/c)$ limit and additionally includes gravitational redshift. Although the numerical implementation of this equation has to solve other problems (for example, the radius dependent range of valid impact parameters $b = [r, \infty]$), the exploration of this idea reveals the physical interpretation of the conservation laws found in the first and second part: The conserved quantities are those measured by an observer at infinity, but expressed in terms of local quantities measured by comoving observers.

After Wilson's [38] pioneering work, years of code development by several groups (Mezzacappa and Bruenn [33], Yamada et al. [31], Burrows et al. [39], Liebendörfer [26], Rampp [40], Messer [41]) have led to Boltzmann solvers for neutrino transport in spherical symmetry. Dynamical simulations of core collapse supernovae have been presented in the Newtonian limit by Mezzacappa et al. [42], and Rampp and Janka [43]. As general relativistic effects significantly influence the postbounce evolution of a collapsed star (Bruenn et al. [44]), these codes have ultimately to be extended to include full general relativistic radiation hydrodynamics. Such simulations were performed by Liebendörfer et al. [26,27] based on the equations discussed in the first and second part, under omission of neutrino back-reaction. Awareness of the conservation laws is essential in the simulation of supernovae because the observed explosion energy is two orders of magnitude smaller than, for example, the core binding energy or the radiation (neutrino) energy. It has to be excluded that an energy drift, owing to numerical inaccuracy in energy conservation, seriously affects the equalized balance and causes or prevents an explosion.

ACKNOWLEDGMENTS

We thank Jacob Fisker for proofreading the derivations in the Appendix, Jochen Peitz for helpful discussions, and Christian Cardall for comments on the paper. In addition to the support of the Swiss National Science Foundation under contract 20-53798.98, M.L. is supported by the National Science Foundation under contract AST-9877130, A.M. is supported at the Oak Ridge National Laboratory, managed by UT-Battelle, LLC, for the U.S. Department of Energy under contract DE-AC05-00OR22725, and F.-K.T. is supported in part by the Swiss National Science Foundation under contract 20-61822.00 and as a Visiting Distinguished Scientist at the Oak Ridge National Laboratory.

In this Appendix, we derive the Einstein equations in comoving coordinates and spherical symmetry. We follow closely the guidelines of exercise (14.16) in Misner et al. [28]. In spherical symmetry, we are allowed to make the following ansatz for the metric:

$$ds^2 = -e^{2\Phi(a,t)} dt^2 + e^{2\Lambda(a,t)} da^2 + r^2(a,t) (d\vartheta^2 + \sin^2(\vartheta) d\varphi^2).$$

The coordinates consist of an independent time coordinate t and a space coordinate a . The areal radius $r(a, t)$ is chosen such that the area of the two-sphere, $d\vartheta^2 + \sin^2(\vartheta) d\varphi^2$, is $4\pi r^2$. We define the following orthonormal noncoordinate basis

$$\begin{aligned}\omega^t &= e^{\Phi(a,t)} dt \\ \omega^a &= e^{\Lambda(a,t)} da \\ \omega^\vartheta &= r(a,t) d\vartheta \\ \omega^\varphi &= r(a,t) \sin(\vartheta) d\varphi\end{aligned}\tag{A1}$$

and calculate its exterior derivatives. From $0 = d\omega^\mu + \omega^\mu_\nu \wedge \omega^\nu$ (14.31a,b in Misner et al. [28]) one determines the nonvanishing connections

$$\begin{aligned}\omega_{ta} &= -\dot{\Phi} e^{-\Lambda} \omega^t - \dot{\Lambda} e^{-\Phi} \omega^a \\ \omega_{t\vartheta} &= -\frac{\dot{r}}{r} e^{-\Phi} \omega^\vartheta \\ \omega_{t\varphi} &= -\frac{\dot{r}}{r} e^{-\Phi} \omega^\varphi \\ \omega_{a\vartheta} &= -\frac{r'}{r} e^{-\Lambda} \omega^\vartheta \\ \omega_{a\varphi} &= -\frac{r'}{r} e^{-\Lambda} \omega^\varphi \\ \omega_{\vartheta\varphi} &= -\frac{1}{r} \frac{\cos(\vartheta)}{\sin(\vartheta)} \omega^\varphi.\end{aligned}\tag{A2}$$

A dot denotes the derivative with respect to coordinate time t and a prime denotes the derivative with respect to the spatial coordinate a . The exterior derivatives of these connections lead to the curvature tensor, and finally the Einstein tensor:

$$\begin{aligned}G^t_t &= -(F + 2\overline{F}) \\ G^t_a &= 2H \\ G^a_a &= -(F + 2\overline{E}) \\ G^\vartheta_\vartheta &= -(\overline{E} + E + \overline{F}) \\ G^\varphi_\varphi &= -(E + \overline{F} + \overline{E}),\end{aligned}$$

where

$$\begin{aligned}E &= (-\Phi'' + \Phi' \Lambda' - \Phi'^2) e^{-2\Lambda} + (\ddot{\Lambda} - \dot{\Phi} \dot{\Lambda} + \dot{\Lambda}^2) e^{-2\Phi} \\ \overline{E} &= \frac{1}{r} \left[(\ddot{r} - \dot{r} \dot{\Phi}) e^{-2\Phi} - r' \Phi' e^{-2\Lambda} \right] \\ H &= \frac{1}{r} (\dot{r}' - \dot{r} \Phi' - r' \dot{\Lambda}) e^{-(\Phi+\Lambda)} \\ F &= \frac{1}{r^2} (1 + \dot{r}^2 e^{-2\Phi} - r'^2 e^{-2\Lambda}) \\ \overline{F} &= \frac{1}{r} \left[(-r'' + r' \Lambda') e^{-2\Lambda} + \dot{r} \dot{\Lambda} e^{-2\Phi} \right].\end{aligned}$$

In the following, we try to write the Einstein equations $G = 8\pi T$ as concise as possible in terms of quantities that are defined in the restframe of a fluid. The nonvanishing components of the stress-energy tensor of a fluid in its

comoving frame are given in Eq. (2). We include an addition, Q , to the diagonal component, representing artificial viscosity as defined in section III, Eq. (24):

$$T^{aa} = p + Q, \quad T^{\vartheta\vartheta} = T^{\varphi\varphi} = p - \frac{1}{2}Q.$$

First, we eliminate the exponentials in the metric by substituting the lapse function α for e^{Φ} and the function r'/Γ for e^{Λ} in order to retrieve notation (1). Then, we have to make sure that the yet unspecified space coordinate a is attached to matter, i. e., that it is a Lagrangian (comoving) coordinate. The rest mass between coordinate a_0 and a_1 at a fixed coordinate time t is given from the metric by

$$A(a_0, a_1) = \int_0^{2\pi} \int_0^{\pi} \int_{a_0}^{a_1} \rho \frac{r'}{\Gamma} da r d\vartheta r \sin(\vartheta) d\varphi = \int_{a_0}^{a_1} \frac{4\pi r^2 r' \rho}{\Gamma} da.$$

We tie the spatial coordinate to rest mass by requiring that $A(a_0, a_1) = \int_{a_0}^{a_1} da$ for arbitrary boundaries a_0, a_1 . This is equivalent to adopting the relation

$$r' = \frac{\Gamma}{4\pi r^2 \rho}. \quad (\text{A3})$$

Further, we define a “velocity” $u = \dot{r}/\alpha$ that describes the change of areal radius with proper time of the comoving observer.

From the nondiagonal component of the Einstein equations we derive in three steps:

$$\begin{aligned} \frac{1}{2}G^{ta} = H &= \frac{\Gamma}{r} \left[\frac{u'}{r'} - \frac{1}{\alpha} \left(\frac{\dot{r}'}{r'} - \frac{\dot{\Gamma}}{\Gamma} \right) \right] = 4\pi q, \\ \frac{1}{\alpha} \left(\frac{\dot{r}'}{r'} - \frac{\dot{\Gamma}}{\Gamma} \right) &= \frac{u'}{r'} - \frac{4\pi r q}{\Gamma}, \end{aligned} \quad (\text{A4})$$

$$\frac{1}{\alpha} \frac{\dot{\Gamma}}{\Gamma} = u \frac{\Phi'}{r'} + \frac{4\pi r q}{\Gamma}. \quad (\text{A5})$$

Equations (A3) and (A4) lead in three further steps to the evolution equation for the rest mass density:

$$\begin{aligned} \frac{\partial}{\alpha \partial t} \left(\frac{1}{\rho} \right) &= \frac{\partial}{\alpha \partial t} \left(\frac{4\pi r^2 r'}{\Gamma} \right) \\ &= \frac{4\pi r^2 r'}{\Gamma} \left(\frac{2u}{r} + \frac{1}{\alpha} \left(\frac{\dot{r}'}{r'} - \frac{\dot{\Gamma}}{\Gamma} \right) \right), \\ &= \frac{1}{\Gamma} \left[(4\pi r^2 u)' - \frac{4\pi r q}{\rho} \right]. \end{aligned} \quad (\text{A6})$$

Eq. (A4) can also be combined with the time-time component of the Einstein equation to give in three more steps:

$$\begin{aligned} \frac{1}{2}G^{tt} = F + 2\bar{F} &= \frac{1}{2r^2} (1 + u^2 - \Gamma^2) - \frac{\Gamma}{r} \left[\frac{\Gamma'}{r'} - \frac{u}{\alpha\Gamma} \left(\frac{\dot{r}'}{r'} - \frac{\dot{\Gamma}}{\Gamma} \right) \right], \\ \frac{1}{2}r^2 r' G^{tt} &= \frac{r'}{2} (1 + u^2 - \Gamma^2) + r r' \left[\frac{\Gamma\Gamma'}{r'} - \frac{u u'}{r'} - \frac{4\pi r u q}{\Gamma} \right], \\ \left[\frac{r}{2} (1 + u^2 - \Gamma^2) \right]' &= 4\pi r^2 r' \rho (1 + e) + \frac{4\pi r^2 r' u q}{\Gamma}. \end{aligned} \quad (\text{A7})$$

It is now convenient to define a “gravitational mass” $m = \frac{r}{2} (1 + u^2 - \Gamma^2)$ that leads, with the help of relation (A3) and (A7), to the concise expressions

$$\Gamma = \sqrt{1 + u^2 - \frac{2m}{r}} \quad (\text{A8})$$

$$m' = \Gamma (1 + e) + \frac{u q}{\rho}. \quad (\text{A9})$$

The physical interpretation of $m(a_1, t)$ as the total energy of the fluid inside coordinate a_1 becomes clear if one looks at an evolution equation for the mass m . In order to derive such an equation, we take the time derivative of Eq. (A8). We already have the time derivative of Γ in Eq. (A5) and the time derivative of r in the definition of the velocity, but we still need to calculate the time derivative of the velocity before being able to isolate the time derivative of m . An evolution equation for the velocity can be deduced from the space-space component of the Einstein equations:

$$\begin{aligned}\frac{1}{2}G^{aa} &= -F - 2\bar{E} = -\frac{1}{r^2}\frac{m}{r} - \frac{1}{r}\left(\frac{\dot{u}}{\alpha} - \Gamma^2\frac{\Phi'}{r'}\right) = 4\pi(p + Q), \\ \frac{\dot{u}}{\alpha} &= \Gamma^2\frac{\Phi'}{r'} - \frac{m}{r^2} - 4\pi r(p + Q).\end{aligned}\tag{A10}$$

Therefore, the evolution of the gravitational mass is

$$\frac{\partial m}{\alpha \partial t} = -4\pi r^2(u(p + Q) + \Gamma q).\tag{A11}$$

The change of the total energy m within a sphere is given by surface work against the pressure and the in/out flow of energy by transport processes.

Analogous to the derivation of the density evolution equation (A6) it is possible to take the time derivative of Eq. (A9) and use Eq. (A11) together with one of the angular components of the Einstein equations to derive an evolution equation for the specific energy. However, we choose to derive these equations more simply from the vanishing four-divergence of the stress-energy tensor. First, we list out the nonvanishing connection coefficients from (A2) and the definition $\omega_{\mu\nu} = \Gamma_{\mu\nu\alpha}\omega^\alpha$ that are used in the following computation of the four-divergence:

$$\begin{aligned}\Gamma\frac{\Phi'}{r'} &= \Gamma_{at}^t = \Gamma_{tt}^a \\ \rho\frac{\partial}{\alpha\partial t}\left(\frac{1}{\rho}\right) - \frac{2u}{r} &= \Gamma_{aa}^t = \Gamma_{ta}^a \\ \frac{u}{r} &= \Gamma_{\vartheta\vartheta}^t = \Gamma_{t\vartheta}^\vartheta = \Gamma_{\varphi\varphi}^t = \Gamma_{t\varphi}^\varphi \\ \frac{\Gamma}{r} &= -\Gamma_{\vartheta\vartheta}^a = \Gamma_{a\vartheta}^\vartheta = -\Gamma_{\varphi\varphi}^a = \Gamma_{a\varphi}^\varphi \\ \frac{1}{r}\frac{\cos\vartheta}{\sin\vartheta} &= -\Gamma_{\varphi\varphi}^\vartheta = \Gamma_{\vartheta\varphi}^\varphi.\end{aligned}$$

Note that we have to respect $\Gamma_{\mu\nu}^\alpha \neq \Gamma_{\nu\mu}^\alpha$ in our noncoordinate basis (A1). With this in mind, one easily obtains an evolution equation for the specific energy from the time component of the four-divergence:

$$\begin{aligned}0 &= \frac{1}{\rho}T^{t\nu}_{;\nu} \\ &= \frac{\partial e}{\alpha\partial t} + (p + Q)\frac{\partial}{\alpha\partial t}\left(\frac{1}{\rho}\right) + \frac{1}{\alpha^2}(4\pi r^2 q \alpha^2)' - \frac{3u}{r}\frac{Q}{\rho}.\end{aligned}\tag{A12}$$

The last equation - usually used to determine Φ and the lapse function $\alpha = e^\Phi$ - is then derived from the space component of the four-divergence:

$$\begin{aligned}0 &= \frac{r'}{\Gamma}\frac{1}{\rho}T^{a\nu}_{;\nu} \\ &= \frac{(p + Q)'}{\rho} + \left(1 + e + \frac{p + Q}{\rho}\right)\Phi' + \frac{\partial}{\alpha\partial t}\left(\frac{1}{4\pi r^2\rho}\frac{q}{\rho}\right) + \frac{3r'}{r}\frac{Q}{\rho}.\end{aligned}\tag{A13}$$

This set of equations (without the artificial viscosity) was, with some approximations, derived by Misner and Sharp [4,34].

[1] S. A. Colgate and M. H. Johnson, Physical Review Letters **5**, 235 (1960).

- [2] S. A. Colgate and R. H. White, *Astrophysical Journal* **143**, 626 (1966).
- [3] M. M. May and R. H. White, *Computational Physics* **7**, 219 (1967).
- [4] C. W. Misner and D. H. Sharp, *Physical Review* **136**, B571 (1964).
- [5] R. W. Lindquist, *Annals of Physics* **37**, 487 (1966).
- [6] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An introduction to current research*, edited by L. Witten (Wiley, New York, 1962).
- [7] L. Smarr and J. W. York, *Physical Review D* **17**, 2529 (1978).
- [8] A. Mezzacappa and R. A. Matzner, *Astrophysical Journal* **343**, 853 (1989).
- [9] D. Mihalas, in *Astrophysical radiation hydrodynamics*, edited by K.-H. Winkler and M. L. Norman (NATO ASI Series C **188**, 1986).
- [10] A. Mezzacappa and S. W. Bruenn, *Astrophysical Journal* **405**, 637 (1993).
- [11] P. J. Schinder and S. A. Bludman, *Astrophysical Journal* **346**, 350 (1989).
- [12] J. R. Wilson, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, 1979).
- [13] S. L. Shapiro and S. A. Teukolsky, *Astrophysical Journal* **235**, 199 (1980).
- [14] J. M. Bardeen and T. Piran, *Physics Reports* **96**, 205 (1983).
- [15] J. V. Romero, J. M.^a Ibanez, J. M.^a Marti, and J. A. Miralles, *Astrophysical Journal* **462**, 839 (1996).
- [16] P. J. Schinder, S. A. Bludman, and T. Piran, *Physical Review D* **37**, 2722 (1988).
- [17] W. C. Hernandez and C. W. Misner, *Astrophysical Journal* **143**, 452 (1966).
- [18] T. W. Baumgarte, S. L. Shapiro, and S. A. Teukolsky, *Astrophysical Journal* **443**, 717 (1995).
- [19] L. I. Petrich, S. L. Shapiro, and S. A. Teukolsky, *Physical Review D* **33**, 2100 (1986).
- [20] M. Liebendörfer and F.-K. Thielemann, in *Nineteenth Texas Symposium on Relativistic Astrophysics*, edited by E. Aubourg, T. Montmerle, J. Paul, and P. Peter (Elsevier Science B. V., Amsterdam, 2000).
- [21] K. A. Van Riper, *Astrophysical Journal* **232**, 558 (1979).
- [22] S. W. Bruenn, *Astrophysical Journal Supplement* **58**, 771 (1985).
- [23] F. D. Swesty, *Astrophysical Journal* **445**, 811 (1995).
- [24] S. Yamada, *Astrophysical Journal* **475**, 720 (1997).
- [25] S. F. Davis, *SIAM J. Sci. Stat. Comput.* **9**, 445 (1988).
- [26] M. Liebendörfer, Ph.D. thesis, University of Basel, (Basel, Switzerland, 2000).
- [27] M. Liebendörfer, A. Mezzacappa, F.-K. Thielemann, O. E. B. Messer, R. W. Hix, and S. W. Bruenn, *Physical Review D*, submitted (2000).
- [28] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman and Company, San Francisco, 1973).
- [29] J. Von Neumann and R. D. Richtmyer, *J. Appl. Phys.* **21**, 232 (1950).
- [30] W. M. Tscharnutter and K.-H. Winkler, *Comput. Phys. Communications* **18**, 171 (1979).
- [31] S. Yamada, H.-T. Janka, and H. Suzuki, *Astronomy and Astrophysics* **344**, 533 (1999).
- [32] J. I. Castor, *Astrophysical Journal* **178**, 779 (1972).
- [33] A. Mezzacappa and S. W. Bruenn, *Astrophysical Journal* **405**, 669 (1993).
- [34] C. W. Misner and D. H. Sharp, *Phys. Lett.* **15**, 279 (1965).
- [35] K.-H. A. Winkler, M. L. Norman, and D. Mihalas, *Quant. Spectrosc. Transfer* **31**, 473 (1984).
- [36] S. K. Godunov, *Mat. Sbornik* **47**, 271 (1959).
- [37] J. M.^a Marti and E. Müller, the electronic journal *Living Reviews in Relativity* **2**, (1999).
- [38] J. R. Wilson, *Astrophysical Journal* **163**, 209 (1971).
- [39] A. Burrows, T. Young, P. A. Pinto, R. Eastman, and T. Thompson, *Astrophysical Journal* **539**, 865 (2000).
- [40] M. Rampp, Ph.D. thesis, TU München, (München, Germany, 2000).
- [41] O. E. Messer, Ph.D. thesis, University of Tennessee, (Knoxville, USA, 2000).
- [42] A. Mezzacappa, M. Liebendörfer, O. E. B. Messer, R. W. Hix, F.-K. Thielemann, and S. W. Bruenn, *Physical Review Letters*, submitted (2000).
- [43] M. Rampp and H.-T. Janka, *Astrophysical Journal Letters* **539**, L33 (2000).
- [44] S. W. Bruenn, K. R. DeNisco, and A. Mezzacappa, *Astrophysical Journal*, submitted (2000).